Quantum algorithms for classification

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{8,9}-11-2018, Nancy, Journées Informatique Quantique

1. What are we doing here?

2. Toolbox

3. Quantum algorithm for classification..
   - QSFA
   - QFDC
   - q-means

4. ..on real data

Unsupervised methods

$$X \in \mathbb{R}^{n \times d}$$

- Anomaly detection
- Clustering
- Blind signal separation
- Text mining

Supervised methods

$$X \in \mathbb{R}^{n \times d}, Y \in \mathbb{R}^{n \times m}$$

- Regression
- Pattern recognition
- Time series forecasting
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[HHL09] ...

Figure: A distressing meme [1]. Ewin Tang employing shock and awe tactics against the quantum community: http://arxiv.org/abs/1811.00414

[1] Quantum Computing Memes for QMA-Complete Teens, Journal of Memeological Depression and Anxiety (2018).

QML team @ IRIF

- Iordanis Kerenidis
- Jonas Landman
- Anupam Prakash

1 - QRAM

Let $X \in \mathbb{R}^{n \times d}$. There is a quantum algorithm that

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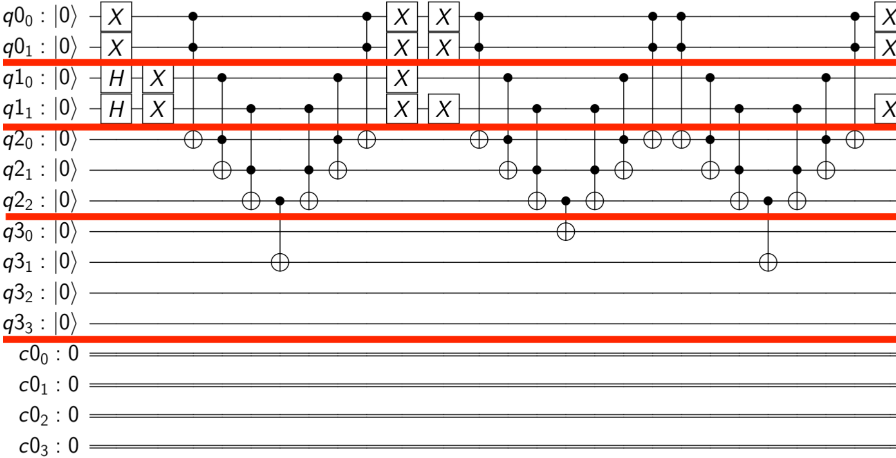
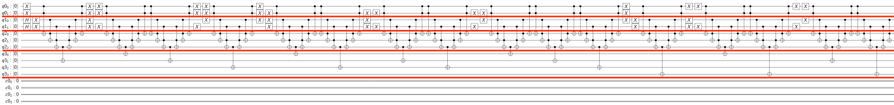


q0_0 : |0⟩
q0_1 : |0⟩
q1_0 : |0⟩
q1_1 : |0⟩
q2_0 : |0⟩
q2_1 : |0⟩
q2_2 : |0⟩
q3_0 : |0⟩
q3_1 : |0⟩
q3_2 : |0⟩
q3_3 : |0⟩
c0_0 : 0
c0_1 : 0
c0_2 : 0
c0_3 : 0
X
H

QRAM [[2,3,4],[5,6,7],[8,9,10]]



2 - Q-BLAS

- $M := \sum_i \sigma_i u_i v_i^T \in \mathbb{R}^{d\times d}$, $\|M\|_2 = 1$, in QRAM
- $x \in \mathbb{R}^d$ in QRAM.

There is a quantum algorithm that w.h.p. returns :

(i) $|z\rangle$ such that $\left\| |z\rangle - |M^{-1}x\rangle \right\| \leq \epsilon$
in time $\widetilde{O}(\kappa(M)\mu(M)\log(1/\epsilon))$

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(iii) a state $|M^{+}_{\leq\theta,\delta} M_{\leq\theta,\delta} x\rangle$
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Get estimates of $\|z\| = f(M)x$ (with mult. error ϵ_2, time $O() \cdot \epsilon_2^{-1}$)

Gilyén, András, et al. "Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics." arXiv preprint arXiv:1806.01838 (2018).

2.5 - Q-BLAS

- $A := \sum_i \sigma_i u_i v_i^T, \quad B := \sum_i \lambda_i w_i l_i^T \in \mathbb{R}^{d \times d}$ in QRAM
$\|A\|_2 = \|B\|_2 = 1$, in QRAM
- $x \in \mathbb{R}^d$ in QRAM.

There is a quantum algorithm that w.h.p. returns :

(i) $|z\rangle$ such that $\left\| |z\rangle - |(AB)^{-1}x\rangle \right\| \leq \epsilon$

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3 - Compute distances

$V \in \mathbb{R}^{n \times d}, C \in \mathbb{R}^{k \times d}$ in the QRAM, $\Delta > 0$ and $\epsilon > 0$
There is a quantum algorithm that w.h.p. and in time
$\widetilde{O}\left(\frac{(T(V)+T(C))Z\log(1/\Delta)}{\epsilon}\right)$

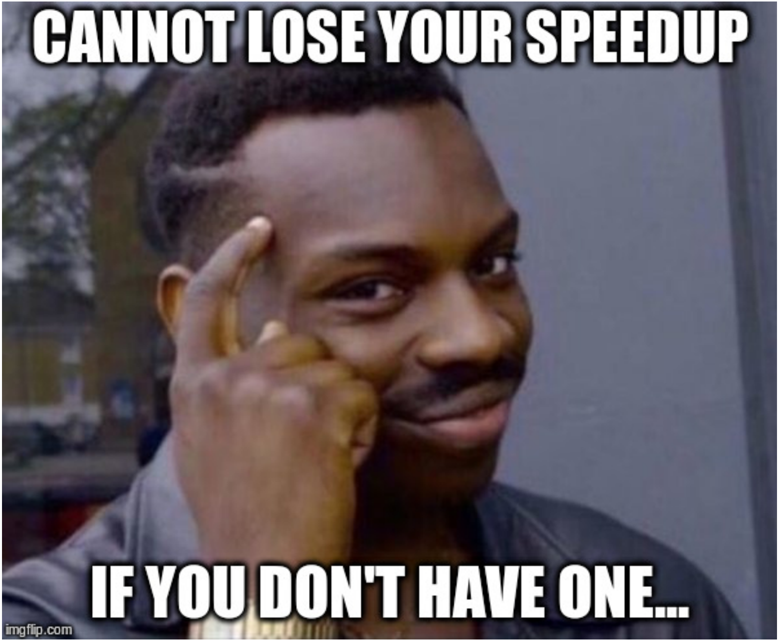
$$|i\rangle\,|j\rangle\,|0\rangle \mapsto \;|i\rangle\,|j\rangle\,|\overline{d(v_i, c_j)}\rangle$$

where $|\overline{d(v_i, c_j)} - d(v_i, c_j)| \leqslant \epsilon$, where
$Z = \max_{i,j}(\|v_i\|^2 + \|c_j\|^2)$.

Based on: Wiebe, N., Kapoor, A., & Svore, K. (2014). Quantum algorithms for nearest-neighbor methods for supervised and unsupervised learning. arXiv preprint arXiv:1401.2142.

Other..

- Amplitude estimation
- Amplitude amplification
- Hamiltonian simulation
- Phase estimation
- Quantum Random Walks
- Swaps


CANNOT LOSE YOUR SPEEDUP
IF YOU DON'T HAVE ONE...
imgflip.com

Slow Feature Analysis (Supervised)

Input signal: $x(i) \in \mathbb{R}^d$. Task: Learn K functions:

$$y(i) = [g_1(x(i)), \cdots, g_K(x(i))]$$

Such that $\forall j \in [K]$. Minimize:

$$\Delta(y_j) = \frac{1}{a} \sum_{k=1}^{K} \sum_{\substack{s,t \in T_k \\ s<t}} \big(g_j(x(s)) - g_j(x(t))\big)^2$$

Constraints on output signal: average of components is 0, variance of components is 1, signals are decorrelated.

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$$AW = BW\Lambda$$

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$Z := X^+ X$ ..now $Z^T Z = I$
Freebie Theorem!
There exists an efficient quantum algorithm for
whitening that return $|Z\rangle := X^+ |X\rangle$

Step 2: projection

$$
\begin{array}{ccc}
X & \xrightarrow{Der.} & \dot{X} \\
\downarrow{\scriptstyle Whit.} & & \downarrow{\scriptstyle Whit.} \\
Z & \xrightarrow{Der.} & \dot{Z}
\end{array}
$$

- Whiten data $|X\rangle \mapsto |Z\rangle$
- Project data in slow feature space $|Z\rangle \mapsto |Y\rangle$

New algo! QSFA

- Let $X = \sum_i \sigma_i u_i v_i^T \in \mathbb{R}^{n \times d}$, $\dot{X} \in \mathbb{R}^{n \log n \times d}$ QRAM.
- Let $\epsilon, \theta, \delta, \eta > 0$.

There exists a quantum algorithm that produces:

- $|\overline{Y}\rangle$ with $|\,|\overline{Y}\rangle - |A^{+}_{\leq\theta,\delta} A_{\leq\theta,\delta} Z\rangle\,| \leq \epsilon$ in time

$$\tilde{O}\Big(\Big(\kappa(X)\mu(X)\log(1/\varepsilon) + \frac{(\kappa(X)+\kappa(\dot{X}))(\mu(X)+\mu(\dot{X}))}{\delta\theta}\Big)$$

$$\dots \times \frac{||Z||}{||A^{+}_{\leq\theta,\delta} A_{\leq\theta,\delta} Z||}\Big)$$

- $\overline{\|Y\|}$ with $|\overline{\|Y\|} - \|Y\|\,| \leq \eta \|Y\|$ with an additional $1/\eta$ factor.

New algo! QFDC (Supervised)

$X_k \in \mathbb{R}^{|T_k| \times d}$ matrix of elements labeled k
$X_0 \in \mathbb{R}^{|T_k| \times d}$ repeats the row x_0 for $|T_k|$ times.

$$F_k(x_0) = \frac{\|X_k - X_0\|_F^2}{2(\|X_k\|_F^2 + \|X_0\|_F^2)},$$

$$\frac{1}{\sqrt{N_k}}\Big(|0\rangle \sum_{i \in T_k} \|x(0)\| \, |i\rangle \, |x(0)\rangle + |1\rangle \sum_{i \in T_k} \|x(i)\| \, |i\rangle \, |x(i)\rangle \Big)$$

$$h(x_0) = min_k\{F_k(y_0) = p(|1\rangle)\}$$

Combining QSFA and QFDC

Require:

X and $\dot{X}$ in QRAM, test vector $x(0)$, and $\varepsilon, \eta > 0$ error param.

Ensure:

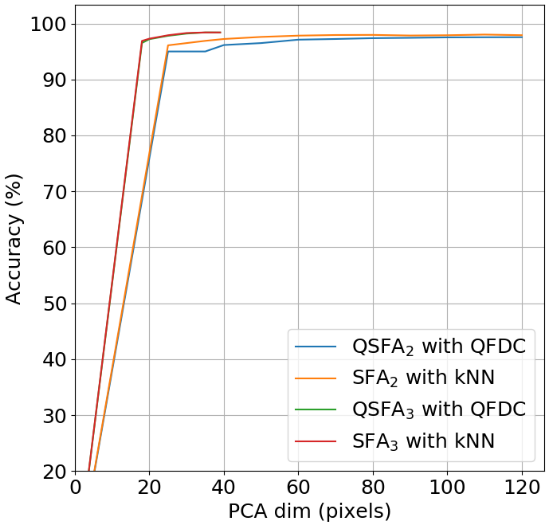
A label for $x(0)$

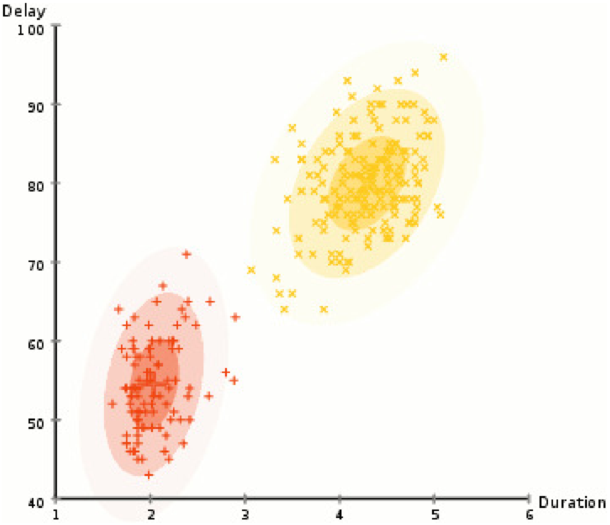
```
for k = 1 → K do
    s_k := 0
    Use QSFA to estimate ||Y_k||_F and the norm ||y(0)|| (to error η).
    for r = O(1/η^2) do
```

$$\frac{1}{\sqrt{N_k}}\left(|0\rangle \sum_{i \in T_k} \|y(0)\| \, |i\rangle \, |y(0)\rangle + |1\rangle \sum_{i \in T_k} \|y(i)\| \, |i\rangle \, |y(i)\rangle \right)$$

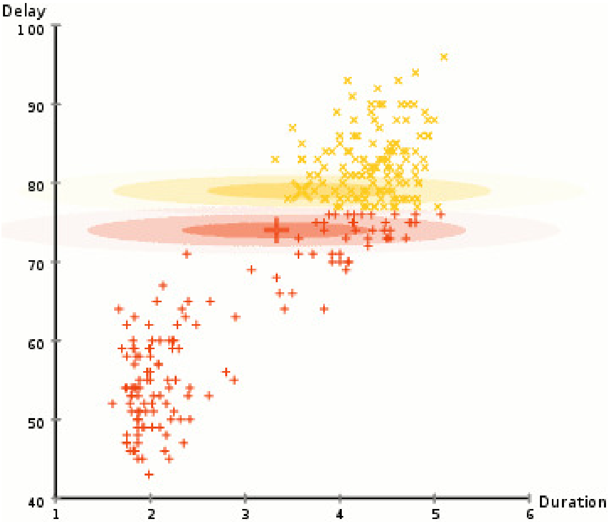
```
        Apply a Hadamard to the first register
        Measure first reg. If |1⟩ then s_k := s_k + 1
    end for
    Estimate F_k(y(0)) := s_k / r (to error O(ε + η)).
end for
```

Accuracy QSFA+QFDC





From Wikipedia



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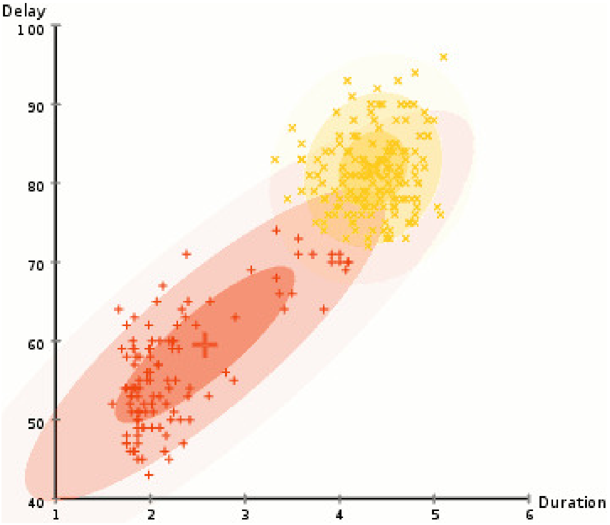

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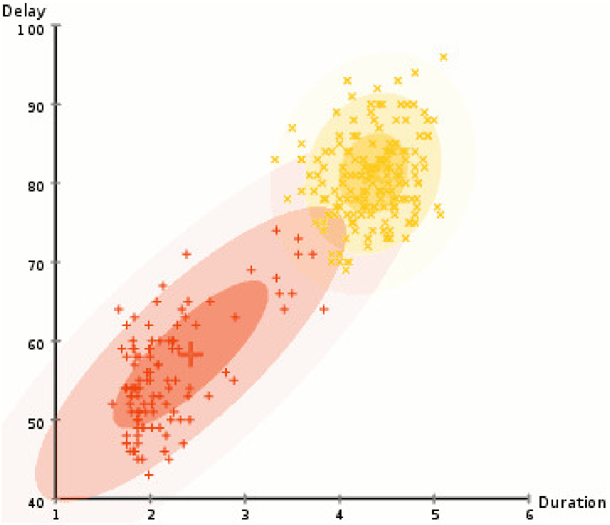

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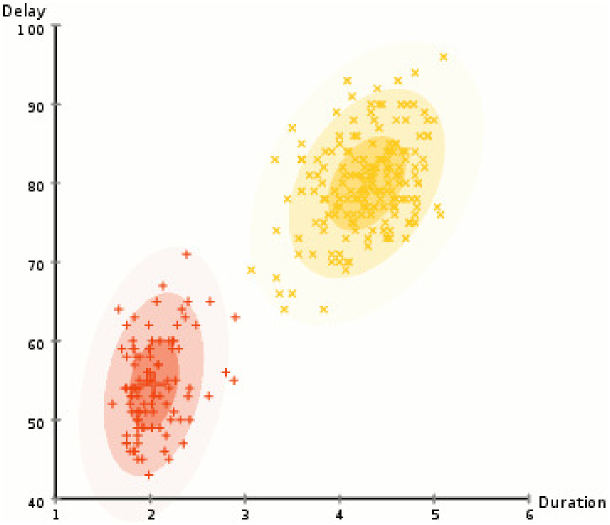

From Wikipedia

Well-clustered data

The data is $(\xi, \beta, \lambda, \eta)$-well clustered if there are $\xi > 0, \quad \beta > 0, \quad 0 \ll \lambda < 1, \quad \eta > 1:$

1. clusters' separation: $d(c_i, c_j) \geq \xi \quad \forall i, j \in [k]$
2. proximity to centroid: A fraction λn of points v_i in the dataset verify: $d(v_i, c_{l(v_i)}) \leq \beta$.
3. dataset's width: All the norms are between 1 and $\eta = \max_i(\|v_i\|)$

k-means (Unsupervised)

Find initial centroids c_j
Repeat until centroids are steady: $|c_j^t - c_j^{t+1} \leq \tau|$

- Calculate distances between all points and all clusters

$$\forall i \in [n], c \in [k] \quad d(v_i, c_i)$$

- Assign points to closer cluster

$$l(v_i) = \arg\min_{c \in [k]} d(v_i, c_i)$$

- Calculate centroids again

$$c_j = \frac{1}{|C_j|} \sum_{i \in C_j} v_i$$

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... is $O(tndk)$:(

q-means (Unsupervised)

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Repeat until centroids are steady: $|c_j^t - c_j^{t+1} \leq \tau|$

- Calculate distances between all points and all clusters

$$\bigotimes_{j=0}^{K} \sum_{i=0}^{n} |i\rangle |j\rangle |d(v_i, c_i)\rangle$$

- Assign points to closer cluster

$$\sum_{i=0}^{n} |i\rangle |l(i)\rangle$$

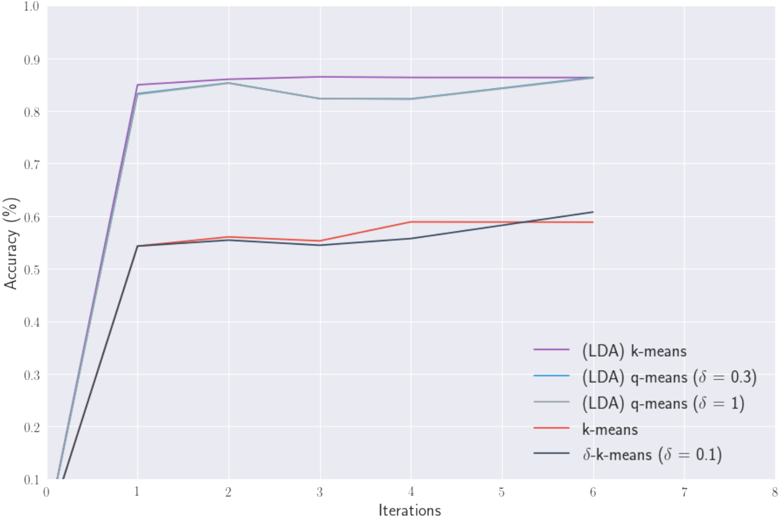
- Calculate centroids again

$$\frac{1}{\sqrt{Z}} \sum_{j \in [k]} \left\| c_j^{t+1} \right\| |c_j^{t+1}\rangle |j\rangle$$

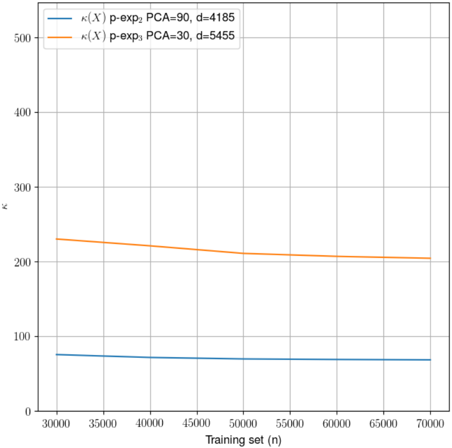
New algo! q-means

For a $(\xi, \beta, \lambda, \eta)$-well clustered dataset $V \in \mathbb{R}^{n \times d}$ in QRAM, there is a quantum algorithm that returns in t steps the k centroids that cluster the dataset consistently with the classical δ-k-means algorithm in time $\widetilde{O}\left(t \cdot \frac{k^2 d Z^{5/2} \kappa(V)}{\delta^3}\right)$.

Accuracy q-means

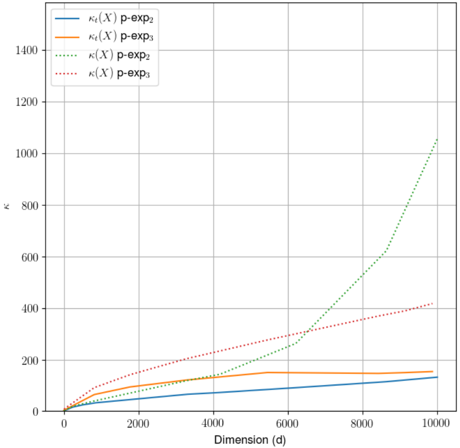


$\lambda_{max}/\lambda_{min}$: more data



Condition number by increasing the number of elements in training set

$\lambda_{max}/\lambda_{min}$: more feature



Condition number by increasing the features (pixels)

#TODOs

- Generalizations…
- Experiments…
- Code…
- New algos…
- Compositions…
- Adversarial QML…
- Privacy preserving QML…

Thanks for your time

there is never enough.
(cit. Dan Geer)

@scinawa

- Quantum Machine Learning ⇒ https://luongo.pro/qml
- QSFA + QFDC ⇒ https://arxiv.org/abs/1805.08837
- q-means ⇒ stay tuned...